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AEC RESEARCH AND DEVELOPMENT REPORT

TID-4500
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NYO-6486
PHYSICS

NOTES ON MAGNETO-HYDRODYNAMICS

VII

FLUID DYNAMICAL ANALOGIES

by

A. A. Blank and Harold Grad

July 15, 1958

Institute of Mathematical Sciences

NEW YORK UNIVERSITY

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PREFACE

The results given in this report are, in large part, identical to those presented in number IV of the original series of mimeographed lecture notes on magneto-hydrodynamics issued in 1954. The class of parallel flows considered in Section 2 of this report has been extended and Section 5 has been added.

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Fluid Dynamical Analogies

A. A. Blank and Harold Grad

1. Introduction

Although the combined fluid and electromagnetic equations generally represent a far more complex system than that of classical fluid dynamics, there are a number of important instances for which the two systems are mathematically identical upon appropriate identification of symbols. The existence of these exact mathematical analogies permits the immediate application of a wealth of known results in fluid dynamics to the relatively new and unexplored field of fluid magnetics. One of these analogies is sufficiently ramified to be treated separately.[1] The results given in this report are misleadingly brief; they allow the application of many known techniques and results to appropriate problems in fluid magnetics.

The fluid dynamical aspect of one of these analogies, i.e. two-dimensional non-steady flow, includes the major portion of the entire literature of fluid dynamics. On the other hand, the mathematically identical fluid magnetic aspect is much more of a special case. This remark illustrates the far greater complexity of the totality of fluid magnetic theory as compared to conventional fluid dynamical theory.

[1] MH-IX, Equilibrium and Force-Free Fields.

The general set of equations which is under consideration is:[2]

$$(1.1) \qquad \frac{d\rho}{dt} + \rho \operatorname{div} u = 0$$

$$(1.2) \qquad \rho \frac{du}{dt} + \operatorname{grad} p = \frac{1}{\mu} \operatorname{curl} B \times B$$

$$(1.3) \qquad \rho \frac{de}{dt} + p \operatorname{div} u = 0, \text{ or } \frac{d\hat{\eta}}{dt} = 0$$

$$(1.4) \qquad \frac{\partial B}{\partial t} - \operatorname{curl} (u \times B) = 0, \operatorname{div} B = 0$$

together with the auxiliary relations

$$(1.5) \qquad E = -u \times B$$

$$(1.6) \qquad J = \frac{1}{\mu} \operatorname{curl} B$$

$$(1.7) \qquad q = \varkappa \operatorname{div} E.$$

We use the notation d/dt to denote the Lagrangian derivative

$$(1.8) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u \cdot \nabla.$$

The sound speed, a_1 is defined by

$$(1.9) \qquad a^2 = \frac{\partial p}{\partial \rho}, \quad p = f(\rho, \hat{\eta})$$

where p is a given function of ρ and of $\hat{\eta}$ (the entropy per unit mass). For an incompressible fluid, we replace (1.1) by

$$(1.10) \qquad \frac{d\rho}{dt} = 0, \quad \operatorname{div} u = 0$$

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[2]See MH-VI, Fluid Magnetic Equations - General Properties.

and ignore (1.3). The fluid is _homogeneous_ incompressible if ρ = constant throughout rather than constant on each particle path. Similarly, if $\hat{\eta}$ = constant throughout the fluid we call the flow _isentropic_, whereas under the condition (1.3) the flow is called _adiabatic_.

In these equations we have assumed infinite conductivity and have omitted displacement current and electrostatic forces.

2. Parallel Flows

The momentum equation (1.2) can be written

$$(2.1) \qquad \rho \frac{\partial u}{\partial t} + \rho(u\cdot\nabla)u + \mathrm{grad}(p + B^2/2\mu) = \frac{1}{\mu}(B\cdot\nabla)B.$$

The form of this equation suggests that we look for steady flows in which B is parallel to u,

$$(2.2) \qquad B = \lambda u;$$

in addition, we consider only an incompressible flow, div u = 0, from which it follows that λ is constant on each streamline (magnetic line)

$$(2.3) \qquad \frac{d\lambda}{dt} = (u\cdot\nabla)\lambda = 0.$$

Since equation (1.4) is satisfied identically, all that remains is to satisfy equation (2.1) which takes the form

$$(2.4) \qquad \rho^*(u\cdot\nabla)u + \mathrm{grad}\ p^* = 0$$

where ρ^* is a constant on each streamline,

$$(2.5) \qquad \rho^* = P - \lambda^2/\mu$$

and p^* is given by

$$(2.6) \qquad p^* = p + B^2/2\mu.$$

We recall that (2.4) together with div u = 0, $d\rho^*/dt = 0$

defines an incompressible steady flow with density ρ^* and pressure p^*.[3] To every such flow, there exists a steady, incompressible, parallel magnetohydrodynamic flow; both ρ and λ can be arbitrarily specified on each streamline in the magnetic analogue. Bernoulli's law holds, viz. the quantity

$$(2.7) \qquad p^* + \frac{1}{2}\rho^* u^2 = p + \frac{1}{2}\rho u^2$$

is constant on each streamline.

The stability of the degenerate case $\rho^* = 0$, $p^* = \text{constant}$, under the homogeneity condition $\rho = \text{constant}$ has been treated by Chandrasekhar (Proc. Nat. Acad. Sci., 42, 273 (1956)).

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[3] It should be remarked that no equation of state is required between p^* and ρ^*.

3. Two-Dimensional Transverse Flows

We consider flows in which all quantities are independent of z but may depend on x, y, and t. The magnetic field is restricted to the z-direction, and the velocity is two-dimensional in the plane (x,y). It is convenient to use two-dimensional vector notation, interpreting the single component of B as a scalar. Equation (1.4) then takes the form

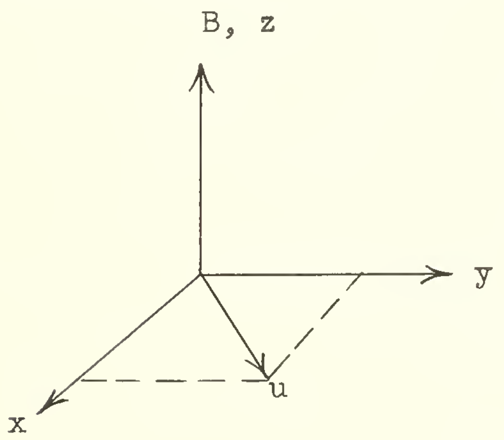

Fig. 1

$$\frac{dB}{dt} + B \operatorname{div} u = 0. \tag{3.1}$$

In other words, conservation of flux takes the same form as conservation of mass in this geometry. The reason is intuitively clear; in a plane flow, the element of area, dxdy, across which the flux is conserved is the same as the two-dimensional "volume" element on which mass is conserved. Equation (3.1) combined with (1.1), yields this result in the form

$$\frac{d}{dt}(B/\rho) = 0; \tag{3.2}$$

the ratio of B to ρ is constant following an element of fluid. The equations (1.2) and (1.3) may be rewritten in terms of the "total pressure"

$$p^* = p + B^2/2\mu \tag{3.3}$$

and "total energy"

$$(3.4) \qquad e^* = e + B^2/2\mu\rho;$$

namely, noting that curl $B \times B = (B\cdot\nabla)B - \nabla(\frac{1}{2}B^2)$ and that $(B\cdot\nabla)B = 0$ in this geometry, we have

$$(3.5) \qquad \rho\frac{du}{dt} + \text{grad } p^* = 0$$

and

$$(3.6) \qquad \rho\frac{de^*}{dt} + p^* \text{ div } u = 0;$$

(the entropy form, $d\hat{\eta}/dt = 0$, can also be used instead of (3.6)).

Despite the fact that equations (3.5) and (3.6) are identical to the ordinary fluid equations with p^* and e^* taking the place of p and e, this is not an exact mathematical analogue because of the absence of a conventional equation of state. More precisely, the complete system,

$$(3.7) \qquad \left\{ \begin{array}{l} \dfrac{d\rho}{dt} + \rho \text{ div } u = 0 \\ \rho\dfrac{du}{dt} + \text{grad } p^* = 0 \\ \dfrac{d\hat{\eta}}{dt} = 0 \\ \dfrac{d\eta^*}{dt} = 0, \qquad \eta^* = B/\rho \end{array} \right.$$

has two quantities, η and $\eta^* = B/\rho$ which behave like entropy, and the "thermodynamics" has three degrees of freedom instead of two; e.g. p^* is a known function of ρ, η, and η^*. An identity which will be found useful below is

(3.8) $$T\,d\eta + \frac{B}{\mu}\,d\eta^* = de^* + p^* d\left(\frac{1}{\rho}\right).$$

The analogy can be made exact by imposing additional restrictions:

(a) Suppose η^* = constant throughout. B can be eliminated from the equation of state (3.3) to obtain

(3.9) $$p^*(\rho, \eta) = p(\rho, \eta) + \frac{1}{2\mu}\,\rho^2\,(\eta^*)^2$$

This equation of state satisfies the usual thermodynamic convexity requirements; in particular we have

(3.10) $$T\,d\eta = de^* + p^* d\left(\frac{1}{\rho}\right)$$

and

(3.11) $$(a^*)^2 = \frac{\partial p^*}{\partial \rho} = a^2 + \frac{1}{\mu}\,\rho(\eta^*)^2 = a^2 + B^2/\mu\rho\ .$$

(b) Suppose η = constant throughout. We interpret η^* as the thermodynamic entropy and obtain the equation of state

(3.12) $$p^*(\rho, \eta^*) = p(\rho) + \frac{1}{2\mu}\,\rho^2(\eta^*)^2.$$

From the relation

(3.13) $$de^* + p^* d\left(\frac{1}{\rho}\right) = \frac{B}{\mu}\,d\eta^*,$$

we see that, by taking

(3.14) $$T^* = B/\mu,$$

we obtain a conventional thermodynamic structure. The sound speed, a^*, is given by the same formula, (3.11), as previously.

(c) Suppose that $\eta^* = f(\eta)$. More precisely, suppose that the curves $\eta(x,y)$ = constant coincide with the curves $\eta^*(x,y)$=constant <u>at the initial instant</u>; we describe this by saying that η^* is a (possibly multi-valued) function of η initially. Since the values of η and η^* are carried with the fluid element, the relation $\eta^* = f(\eta)$ persists for all time. Either η or η^* can be chosen as the entropy; let us take η. The pressure equation of state is

$$(3.15) \qquad p^*(\rho, \eta) = p(\rho, \eta) + \frac{1}{2\mu} \rho^2 (f(\eta))^2 ,$$

the sound speed is as before, and the thermodynamic identity

$$(3.16) \qquad T^* d\eta = de^* + p^* d(\frac{1}{\rho})$$

follows when we take

$$(3.17) \qquad T^* = T + \frac{1}{\mu} \rho \, f(\eta) f'(\eta).$$

(d) In a one-dimensional flow (variables x and t), the relation $\eta^* = f(\eta)$ is satisfied for arbitrary initial data, and the formulation (c) follows.

All conventional two-dimensional gas dynamical theory applies to the above magnetic analogues, provided that no specific assumptions have been made regarding the equation of state. One of the interesting features of these hydromagnetic flows would seem to be the possibility of creating arbitrary equations of state at will by the proper choice of initial configuration. Another way of looking at this is that one can construct a fluid with a sound speed which is an arbitrary function of position much more easily than in a conventional gas.

4. Conservation of Circulation and Bernoulli's Law

Since the previously described analogies are exact, the conventional results on integrals of the motion follow under the same conditions as quoted for the conventional fluids. However, it is illuminating to derive these results directly.

The circulation assigned to a closed curve Γ is defined by

$$(4.1) \qquad C = \oint_{\Gamma} u \cdot dx = \oint_{\Gamma} u \cdot \frac{dx}{d\sigma}\, d\sigma$$

where σ is a parametrization of Γ. For a curve, $\Gamma(t)$, which is moving with the fluid (σ is fixed to a fluid element) we have

$$\frac{dC}{dt} = \oint \frac{du}{dt} \cdot \frac{dx}{d\sigma}\, d\sigma + \oint u \cdot \frac{du}{d\sigma}\, d\sigma$$

$$(4.2) \qquad = \oint \frac{du}{dt} \cdot dx + \oint d(\tfrac{1}{2}u^2) = \oint \frac{du}{dt} \cdot dx$$

$$= \oint \frac{1}{\rho} \left\{ \frac{1}{\mu}(B \cdot \nabla)B - \nabla p^* \right\} \cdot dx$$

This will not be zero and yet admit any significant generality, except in the two-dimensional transverse flows just treated. In that case $(B \cdot \nabla)B = 0$ and

$$(4.3) \qquad \frac{dC}{dt} = -\oint_{\Gamma(t)} \frac{1}{\rho}\, dp^*.$$

Circulation will be conserved if ρ = constant (homogeneous incompressible flow) or if p^* is a function of ρ alone. The

latter will be the case when both η and η^* are constant throughout the flow (cf. equations (3.9), (3.12), and (3.15)). The general condition for conservation of circulation (cf. equation (3.8)) is that a potential, Ω, exist satisfying

$$(4.4)\qquad T d\eta + \frac{B}{\mu} d\eta^* = d(e^* + p^*/\rho) - \frac{1}{\rho} dp^* = d\Omega$$

If the flow is irrotational,

$$(4.5)\qquad u = \nabla\phi$$

and if the potential Ω exists (e.g., if η and η^* are constant), then, using $(u\cdot\nabla)u = \text{curl } u \times u + \nabla(\frac{1}{2}u^2)$, we find that

$$(4.6)\qquad \frac{du}{dt} + \frac{1}{\rho}\nabla p^* = \nabla(\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + h^* - \Omega) = 0$$

or (Bernoulli's law)

$$(4.7)\qquad \frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + h^* - \Omega = a(t)$$

where the "total enthalpy" h^* is

$$(4.8)\qquad h^* = e^* + p^*/\rho.$$

For a <u>steady</u> irrotational flow with a potential Ω, we have

$$(4.9)\qquad \nabla(\frac{1}{2}u^2) + \frac{1}{\rho}\nabla p^* = 0,$$

from which, taking the dot product with u and using $\frac{d}{dt} = u\cdot\nabla$, we obtain

$$(4.10)\qquad \frac{d}{dt}(\frac{1}{2}u^2 + h^* - \Omega) = 0.$$

In this case, $\frac{1}{2}u^2 + h^* - \Omega$ is constant on each streamline.

5. One-Dimensional Transverse Flows

This geometry is, in a sense, the dual of the one treated in Section 3. We assume that all quantities are independent of x and y (the relevant variables are z and t), that B has two components, B_x and B_y, and u has only a z-component. We again have $(B\cdot\nabla)B = 0$, and can write the momentum equation in the form

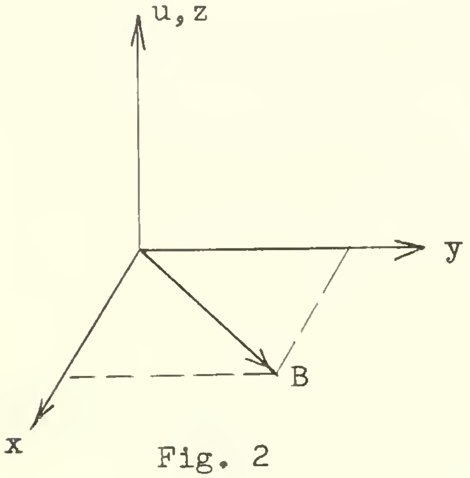

Fig. 2

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p^*}{\partial z} = 0. \tag{5.1}$$

The equation for B, interpreting u as a scalar, is

$$\frac{\partial B}{\partial t} + \frac{\partial}{\partial z}(uB) = 0. \tag{5.2}$$

For the magnitude of B we obtain

$$\frac{\partial}{\partial t}|B| + \frac{\partial}{\partial z}(u|B|) = 0. \tag{5.3}$$

This together with (5.1) and the conservation of mass is a determined system (only the magnitude of B enters in p^*). Moreover, it is exactly the same system that was treated in Section 3, specialized to one space dimension, z, instead of two, (x,y). To complete the solution of the original problem, we note that (5.2) implies that the direction of the vector B is a constant following a fluid element.